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On the specific features of the theory of scattering of two quantum particles by a third massive particle

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Abstract. The behaviour of the solution of the Faddeev equations for the three-body problem if one particle mass goes to infinity has been investigated. It is shown that this solution has no continuous passage to the limit considered. It is also shown that for obtaining the required amplitude of scattering of two light particles by an infinitely heavy particle it is necessary to rearrange the Faddeev equations and to reduce them to the equations of the problem of two particles in a field.

1. Introduction

It is known that many atomic phenomena and nuclear reactions involving heavy ions, as well as the reactions of interaction of quantum particles with a solid, can be considered as three-body problems under the assumption that the mass of one of the particles is infinitely large. It would appear that for the theoretical description of the above-mentioned processes one may use the integral Faddeev equations written down for the case of three particles of finite mass, if one lets the mass of one of the particles go to infinity in these equations. However, as has been demonstrated by Komarov *et al* (1980a, 1981a), such equations totally lose the properties of the original Fredholm system of equations of the three-body problem. For the numerical solution with familiar methods these equations should be rearranged and reduced to the equations of the problem of two particles in an external field.

This result can be explained by the fact that the passage from the problem of three bodies of finite mass to the problem in which one particle becomes infinitely heavy is not continuous.

Firstly, the particle whose mass tends to infinity ceases to be quantum in the limit. This is reflected by the fact that, for example, in the method of secondary quantisation the creation and annihilation operators of such a particle become c numbers.

Secondly, if the mass of one of the particles goes to infinity, in the problem of three bodies with finite masses there is no invariance of three equivalent coordinate systems and, in addition, the law of conservation of momentum ceases to apply.

Thirdly, the Faddeev equations in which the mass of one particle goes to infinity become singular perturbed equations and, as has been shown by Vasil'yeva and Butuzov (1973), can have solutions which do not satisfy the given boundary conditions. It is worth noting that similar singularities arise in the Faddeev equations in the case of the Coulomb interaction between particles.

However, as will be demonstrated below, in contrast to the Coulomb three-body problem the Faddeev equations for the scattering of two light particles by a third massive particle can be rearranged and reduced to numerically solvable equations of the problem of two bodies in an external field.

2. The form of singularities in the Faddeev equations for the case of scattering of two particles by a massive particle

Consider the problem of scattering of three particles, one of which has an infinitely large mass. Let us write the Hamiltonian of this three-body problem in the momentum representation, having chosen as independent variables the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of light particles 1 and 2 with respect to a third massive particle:

$$H(\boldsymbol{p}_1,\boldsymbol{p}_2)=H_0(\boldsymbol{p}_1,\boldsymbol{p}_2)+V_{13}(\boldsymbol{p}_1)+V_{23}(\boldsymbol{p}_2)+V_{12}(\boldsymbol{p}_1-\boldsymbol{p}_2)$$
$$H_0(\boldsymbol{p}_1,\boldsymbol{p}_2)=p_1^2/2m_1+p_2^2/2m_2+(\boldsymbol{p}_1\boldsymbol{p}_2)/M_3. \tag{1}$$

Here m_1, m_2 and M_3 are the masses of particles 1, 2 and 3, and V_{ij} are the potentials of interaction between the particles (subsequently $V_{13}\equiv V_1$, $V_{23}\equiv V_2$).

For determining the scattering operator $T(E)$ of this system there is a Lippmann–Schwinger equation

$$T(E)=(V_1+V_2+V_{12})+(V_1+V_2+V_{12})\mathscr{G}_0(E)T(E) \tag{2}$$

where $\mathscr{G}_0(E)=(E-H_0+\mathrm{i}0)^{-1}$, and E is the total energy.

The solution of the integral equation for the corresponding matrix element of the scattering operator

$$T(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$$

involves in the present case the amplitudes of uncoupled processes, in other words, the function $T(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$ has singular terms which will be designated as $T'(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$. Here $(\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$ and $(\boldsymbol{p}_1,\boldsymbol{p}_2;E)$ are the momenta of the particles and the total energy, respectively, in the final and initial states. Then, obviously,

$$T(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)=T'(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)+\tilde{T}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E) \tag{3}$$

where $\tilde{T}(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$ is the amplitude of the fully coupled processes. It is thus clear that equation (2), written in the momentum representation for the function $T(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$, contains the implicit singular function $T'(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$. The rearrangement of the Lippmann–Schwinger equation, as proposed by Faddeev, was intended to single out the function T' and to obtain equations for the amplitude $\tilde{T}$ (Faddeev 1963). Indeed, the passage from the operator T to the sum of the form

$$T(E)=T_1(E)+T_2(E)+T_{12}(E) \tag{4}$$

where for the terms on the right-hand side of equation (4) there is a system of equations

$$T_1=t_1+t_1\mathscr{G}_0(T_2+T_{12}) \qquad T_2=t_2+t_2\mathscr{G}_0(T_1+T_{12}) \qquad T_{12}=t_{12}+t_{12}\mathscr{G}_0(T_1+T_2) \tag{5}$$

made it possible to single out the amplitudes of uncoupled processes in the problem of three bodies with finite masses explicitly. These amplitudes are the matrix elements of the free terms in the system (5).

However, in the case $M_3 \to \infty$, in the system there is one more uncoupled process corresponding to the independent scattering of particles 1 and 2 by massive particle 3. The scattering amplitude of such a process on the energy shell is of the form

$$-2\pi i\delta(E_1 - E_1')\langle \boldsymbol{p}_1|t_1(E_1)|\boldsymbol{p}_1'\rangle\langle \boldsymbol{p}_2|t_2(E-E_1)|\boldsymbol{p}_2'\rangle \tag{6}$$

where E_1 is the energy of particle 1.

Obviously, in equations (5), written in the momentum representation, this amplitude (6) is implicit and cannot be singled out by the iteration method. It thus appears that it is impossible to solve equations of the form (5) numerically for the case $M_3 \to \infty$. Equations (5) ought to be rearranged so as to single out the amplitude of the uncoupled process just mentioned.

It will be shown that equations (5) for $M_3 \to \infty$ do contain singular terms. For this purpose, consider the special case $V_{12} = 0$. With this assumption, equations (5) in the momentum representation are transformed to

$$\begin{aligned}
&T_1(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E) \\
&\qquad = (2\pi)^3\delta(\boldsymbol{p}_2 - \boldsymbol{p}_2')t_1(\boldsymbol{p}_1, \boldsymbol{p}_1'; E - p_2^2/2m_2) \\
&\qquad\quad + \int \frac{d^3\boldsymbol{p}_1''}{(2\pi)^3} \frac{t_1(\boldsymbol{p}_1, \boldsymbol{p}_1''; E - p_2^2/2m_2)T_2(\boldsymbol{p}_1'', \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E)}{E - p_1''^2/2m_1 - p_2^2/2m_2 + i0} \\
&T_2(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E) \\
&\qquad = (2\pi)^3\delta(\boldsymbol{p}_1 - \boldsymbol{p}_1')t_2(\boldsymbol{p}_2, \boldsymbol{p}_2'; E - p_1^2/2m_1) \\
&\qquad\quad + \int \frac{d^2\boldsymbol{p}_2''}{(2\pi)^3} \frac{t_2(\boldsymbol{p}_2, \boldsymbol{p}_2''; E - p_1^2/2m_1)T_1(\boldsymbol{p}_1, \boldsymbol{p}_2''; \boldsymbol{p}_1', \boldsymbol{p}_2'; E)}{E - p_1^2/2m_1 - p_2''^2/2m_2 + i0}.
\end{aligned} \tag{7}$$

Assume that the energy is related to the momenta of the particles in the final state by the relation

$$E = p_1'^2/2m_1 + p_2'^2/2m_2. \tag{8}$$

Noting that the sum of the amplitudes $T_1 + T_2$ on the energy shell must be a singular function, assume that

$$\begin{aligned}
&T_1(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E) \\
&\qquad = (2\pi)^3\delta(\boldsymbol{p}_2 - \boldsymbol{p}_2')t_1(\boldsymbol{p}_1, \boldsymbol{p}_1'; E - p_2^2/2m_2) \\
&\qquad\quad + \Gamma_1(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E)(p_2'^2/2m_2 - p_2^2/2m_2 + i0)^{-1} \\
&T_2(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E) \\
&\qquad = (2\pi)^3\delta(\boldsymbol{p}_1 - \boldsymbol{p}_1')t_2(\boldsymbol{p}_2, \boldsymbol{p}_2'; E - p_1^2/2m_1) \\
&\qquad\quad + \Gamma_2(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E)(p_1'^2/2m_1 - p_1^2/2m_1 + i0)^{-1}.
\end{aligned} \tag{9}$$

Substituting (9) into (7) yields for Γ_1 and Γ_2 a system of integral equations of the form

$$\begin{aligned}
&\Gamma_1(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E) \\
&\qquad = t_1(\boldsymbol{p}_1', \boldsymbol{p}_1'; E - p_2^2/2m_2)t_2(\boldsymbol{p}_2, \boldsymbol{p}_2'; p_2'^2/2m_2) + \left(\frac{p_2'^2}{2m_2} - \frac{p_2^2}{2m_2}\right) \\
&\qquad\quad \times \int \frac{d^3\boldsymbol{p}_1''}{(2\pi)^3} \frac{t_1(p_1, \boldsymbol{p}_1''; E - p_2^2/2m_2)\Gamma_2(\boldsymbol{p}_1'', \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2'; E)}{(E - p_1''^2/2m_1 - p_2^2/2m_2 + i0)(p_1'^2/2m_1 - p_1''^2/2m_1 + i0)}
\end{aligned}$$

$$\Gamma_2(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$$
$$= t_2(\boldsymbol{p}_2,\boldsymbol{p}'_2;E-p_1^2/2m_1)t_1(\boldsymbol{p}_1,\boldsymbol{p}'_1;p_1'^2/2m_1)+\left(\frac{p_1'^2}{2m_1}-\frac{p_1^2}{2m_1}\right)$$
$$\times\int\frac{d^3p''_2}{(2\pi)^3}\frac{t_2(\boldsymbol{p}_2,\boldsymbol{p}''_2;E-p_1^2/2m_1)\Gamma_1(\boldsymbol{p}_1,\boldsymbol{p}''_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)}{(E-p_1^2/2m_1-p_2''^2/2m_2+\mathrm{i}0)(p_2'^2/2m_2-p_2''^2/2m_2+\mathrm{i}0)}. \quad (10)$$

If we now make use of the familiar relation

$$\frac{t(\boldsymbol{p},\boldsymbol{p}';z)-t(\boldsymbol{p},\boldsymbol{p}';z')}{z-z'}=-\int\frac{d^3p''}{(2\pi)^3}\frac{t(\boldsymbol{p},\boldsymbol{p}'';z)t(\boldsymbol{p}'',\boldsymbol{p}';z')}{(z-p''^2/2\mu+\mathrm{i}0)(z'-p''^2/2\mu+\mathrm{i}0)} \quad (11)$$

then we can show that

$$\Gamma_1(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)=\Gamma_2(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)=t_1(\boldsymbol{p}_1,\boldsymbol{p}'_1,p_1'^2/2m_1)t_2(\boldsymbol{p}_2,\boldsymbol{p}'_2;p'_2/2m_2). \quad (12)$$

Obviously, on the energy shell at $E=p_1^2/2m_2+p_2^2/2m_2=p_1'^2/2m_1+p_2'^2/2m_2$ we have from (9) and (12)

$$T(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}'_1,\boldsymbol{p}'_2;E)$$
$$=(2\pi)^3\delta(\boldsymbol{p}_2-\boldsymbol{p}'_2)t_1(\boldsymbol{p}_1,\boldsymbol{p}'_1;p_1'^2/2m_1)$$
$$+(2\pi)^3\delta(\boldsymbol{p}_1-\boldsymbol{p}'_1)t_2(\boldsymbol{p}_2,\boldsymbol{p}'_2;p_2'^2/2m_2)$$
$$-2\pi\mathrm{i}\delta(p_1^2/2m_1-p_1'^2/2m_1)t_1(\boldsymbol{p}_1,\boldsymbol{p}'_1;p_1'^2/2m_1)t_2(\boldsymbol{p}_2,\boldsymbol{p}'_2;p_2'^2/2m_2). \quad (13)$$

Thus, the solution of the form (9) is the required physical solution. By direct substitution it is easy to show that the functions (9) satisfy both the non-homogeneous equation (5) and the corresponding homogeneous equation to within the terms equal to zero in terms of generalised functions. It is this statement that was proved earlier by Komarov *et al* (1980a) for the equations, similar to (10), written for wavefunctions.

The result obtained above indicates vividly that the integral Faddeev equations for three particles of finite mass, which have the physical solution in terms of classical Hölder functions, when one particle mass goes to infinity are transformed to singular equations of the form (7), which have the required physical solutions of the form (9) in terms of generalised functions.

3. Manifestation of singularities in the Faddeev equations for the case of scattering of two particles by a massive particle

In § 2 it was shown that the singular amplitude, which is implicit in equations (5) for $M_3\to\infty$, is the solution of the system (10) obtained from (5) at $V_{12}=0$. This result is a heuristic consideration for determining the method of rearrangement of the Lippmann–Schwinger equation (2) for $M_3\to\infty$ in order to single out the amplitudes of all uncoupled processes explicitly in the free terms of equations. Such a method of rearrangement must consist in the following (Komarov *et al* 1980b). The total scattering operator $T(E)$ in the present case ($M_3\to\infty$) must be expressed by two terms

$$T(E)=T_{(1,2)}(E)+T_{12}(E). \quad (14)$$

Substituting (14) into the Lippmann–Schwinger equation shows that the first term $T_{(1,2)}(E)$ must have as a free term and kernel the scattering operator for the system of two light particles in the field of massive particle 3 at $V_{12}=0$

$$T_{(1,2)}=\mathcal{T}_{12}+\mathcal{T}_{12}\mathcal{G}_0 T_{12}. \tag{15}$$

The second term must satisfy the equation

$$T_{12}=t_{12}+t_{12}\mathcal{G}_0 T_{(1,2)}. \tag{16}$$

In equation (15), the operator $\mathcal{T}_{12}$ in turn satisfies the equation

$$\mathcal{T}_{12}=(V_1+V_2)+(V_1+V_2)\mathcal{G}_0\mathcal{T}_{12}.$$

The solution of this equation can be found if the operator $\mathcal{T}_{12}(E)$ is expressed in terms of the corresponding total Green function $\mathcal{G}_{(1,2)}(E)$. Since for the system of two independently scattered particles the relation

$$\mathcal{G}_{(1,2)}(E)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\varepsilon}{(-2\pi\mathrm{i})}g_1(\varepsilon+\mathrm{i}0)g_2(E-\varepsilon+\mathrm{i}0)$$

holds, where

$$g_i(z)=g_{i0}(z)+g_{i0}(z)t_i(z)g_{i0}(z),$$

then we have

$$\mathcal{T}_{12}(E)=\mathcal{G}_0^{-1}(E)\left(\int_{-\infty}^{\infty}\frac{\mathrm{d}\varepsilon}{(-2\pi\mathrm{i})}g_1(\varepsilon+\mathrm{i}0)g_2(E-\varepsilon+\mathrm{i}0)\right)\mathcal{G}_0^{-1}(E)-\mathcal{G}_0^{-1}(E).$$

Calculating the matrix elements for the operators $\mathcal{T}_{12}$ and t_{12}, we obtain the expression for the amplitudes of all uncoupled processes taking place in the present problem.

The equations for the scattering of two particles when $V_{12}\neq 0$ are written and examined by Komarov *et al* (1980b, 1981b).

4. Conclusions

We now show that the scattering theory results obtained above should be taken into account in analysing the concrete processes of interaction of a few particles. Consider, for example, the breakup reaction of a coupled system of two particles 1 and 3 of mass (m_1+M_3) under the action of incident particle 2 of mass m_2. Assume that the mass M_3 of particle 3 can be infinitely large. Similar problems are encountered in atomic physics. In particular, in the scattering of electrons by hydrogen-like ions the nuclear mass is usually taken to be infinitely large and the interaction of two electrons at high energies is described in the first Born approximation.

Within the framework of the theory of scattering of two particles in a field the amplitude of the given process at high energy, found from the first iteration of the corresponding integral equations (Komarov *et al* 1980b) and equivalent to the amplitude calculated in the first Born approximation, has the form

$$T\simeq\langle\varphi_{01}\boldsymbol{p}_{02}|V_{12}|\boldsymbol{p}_1\boldsymbol{p}_2\rangle. \tag{17}$$

Here φ_{01} is the wavefunction for the bound state of particle 1 in the field, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are the momenta of the two particles in the final state, and $\boldsymbol{p}_{02}$ is the momentum of particle 2 in its initial state.

Obviously, when $V_{12}=0$, the amplitude T, determined by expression (17), is zero, which is consistent with the familiar result from the general theory of the S matrix for the case of independent processes.

Now consider these processes as the three-body problem for $M_3 \to \infty$. Then the expression for the amplitude of the given reaction in the Born approximation must correspond to the first iterations of the Faddeev equations (Faddeev 1963) and have the following representation:

$$T \simeq \langle \varphi_{01}\boldsymbol{p}_{02}|V_{12}|\boldsymbol{p}_1\boldsymbol{p}_2\rangle + \langle \varphi_{01}\boldsymbol{p}_{02}|V_{23}|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle. \tag{18}$$

The first term in (18) is fully identical with expression (17) and the second term corresponds to the contribution to the amplitude of the process in which the energy and momentum transfer is effected with the aid of a third heavy particle. If we now put $V_{12}=0$, the first term in (18) becomes zero and the second remains non-zero. This discrepancy between the representations for the amplitude of scattering of two particles by a massive particle, written in two different approaches, is explained by the fact that the solution of the Faddeev equations for three particles of finite mass, found by the iteration method, does not go continuously into the solution of equations obtained for $M_3 \to \infty$. Consequently, the representation (18) for the amplitude of the process under consideration cannot be used in the analysis of experimental data if $M_3 \to \infty$. We have given consideration to this result in view of the fact that in atomic physics, for describing the processes which reduce to the problem of interaction of two particles on a massive particle, investigators sometimes use a representation of the form (18) and even explain some of the observed features of experimental cross sections with the aid of the scattering process involving a heavy particle (the second term in equation (18)).

The second example where the results of the present paper should be taken into account pertains to the problems of scattering of four and more particles. Here, in describing the asymptotic state of particles scattered in two independent subsystems, it becomes necessary to solve equations equivalent to the system (7). As has been demonstrated in § 2, the required physical solution of this system can be found in terms of generalised functions rather than in terms of usual functions.

We would like to emphasise the following. In this paper the reaction $1+2+3$ $1+2+3$ (so-called free–free process) has been considered, as an example. Nevertheless, the conclusions are universal. In fact, by obtaining the amplitudes of reactions with the bound states in the initial or final state we have to project the free–free amplitude upon a suitable subspace and pass to the on-shell limit, but the off-shell amplitudes in the equations can be singular functions. To demonstrate this let us consider the triangle diagram for the elastic scattering $1+(2,3) \to 1+(2,3)$ by $M_3 \to \infty$ and $V_{12}=0$:

$$\begin{aligned} T(\boldsymbol{p}, \boldsymbol{p}'; E) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} & \frac{\mathcal{G}(q)}{E - p^2/2m - q^2/2m + \mathrm{i}0} \\ & \times t_1(\boldsymbol{p}, \boldsymbol{p}'; E - q^2/2m) \frac{\mathcal{G}(q)}{E - p'^2/2m - q^2/2m + \mathrm{i}0} \end{aligned} \tag{19}$$

where $\mathcal{G}(q)$ is the form factor normalised with the condition

$$\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{\mathcal{G}^2(q)}{(\varepsilon_0 + q^2/2m)^2} = 1$$

and $\boldsymbol{p}$, $\boldsymbol{p}'$ are initial and final momenta of particle 1. If $E=-\varepsilon_0+(p^2/2m)=-\varepsilon_0+(p'^2/2m)$ then (19) is a well defined function. In other cases the amplitude T has a pole at $p^2=p'^2$. This singularity takes place also in high-order diagrams.

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